# Analytical Bethe Ansatz for open spin chains
# with soliton non preserving boundary conditions

D. Arnaudon[† a], N. Crampé[† b], A. Doikou[† a],

L. Frappat[† a,c] and É. Ragoucy[† a]

[a] *Laboratoire d'Annecy-le-Vieux de Physique Théorique*
*LAPTH, CNRS, UMR 5108, Université de Savoie*
*B.P. 110, F-74941 Annecy-le-Vieux Cedex, France*

[b] *University of York, Department of mathematics*
*Heslington, York YO10 5DD, United Kingdom*

[c] *Member of Institut Universitaire de France*

**Abstract**

We present an "algebraic treatment" of the analytical Bethe ansatz for open spin chains with soliton non preserving (SNP) boundary conditions. For this purpose, we introduce abstract monodromy and transfer matrices which provide an algebraic framework for the analytical Bethe ansatz. It allows us to deal with a generic $gl(\mathcal{N})$ open SNP spin chain possessing on each site an arbitrary representation. As a result, we obtain the Bethe equations in their full generality. The classification of finite dimensional irreducible representations for the twisted Yangians are directly linked to the calculation of the transfer matrix eigenvalues.

[†]arnaudon@lapp.in2p3.fr, nc501@york.ac.uk, doikou@lapp.in2p3.fr, frappat@lapp.in2p3.fr, ragoucy@lapp.in2p3.fr

# Introduction

A great interest for generalizations of integrable quantum spin chains has recently appeared. This renewed interest is primarily due to new applications of these spin chains in various fields, such as condensed matter [1, 2, 3, 4, 5, 6, 7, 8], integrable relativistic quantum field theories [9, 10, 11, 12], quantum chromodynamics theory [13, 14] or AdS/CFT correspondence [15, 16, 17]. In this context, an important development was the definition of integrable quantum spin chain with non-trivial boundaries. Usually, in the framework of spin chain models, these boundaries are characterized by a matricial solution to the reflection equation [18, 19]. However, "new" types of boundaries have been introduced in [20, 21, 22] where the matrix is now solution of the so-called soliton non preserving (SNP) reflection equation. This equation was intensively studied from a mathematical point of view to define the algebras called twisted Yangians [23, 24]. Later, it has been used to introduce boundaries in the affine Toda field theories [25, 26, 27, 28].

In our previous work [29], we used the classification of irreducible finite dimensional representations of the Yangian and of the reflection algebra to construct and solve spin chains where each sites are associated to a different representation of $gl(\mathcal{N})$. In the present paper, we extend this procedure to the SNP case. Indeed, the classification of the representations of the twisted Yangian provided in [30] allows us to obtain the spin chain with non-soliton preserving boundaries where each spin of the chain can be in different representations.

This article is organized as follows. In the first three sections we recall elementary notions on the Yangian of $gl(\mathcal{N})$, the twisted Yangian and the representations of the Yangian. These sections provide all the definitions needed for the article to be self-contained. We also introduce the transfer matrix and determine its symmetry algebra, which is consequently the one of the considered spin chain. In section 4, the classification of the twisted Yangian are described, as well as the fusion procedure, to obtain constraints on the transfer matrix. Using these constraints, we then determine the dressing functions for a general transfer matrix eigenvalue. Finally, we compute the Bethe equations, by analytical Bethe ansatz, for the general SNP $gl(\mathcal{N})$ spin chain.

# 1   Yangian $\mathcal{Y}(gl(\mathcal{N})$

We will consider the $gl(\mathcal{N})$ invariant $R$ matrices [31, 32]

$$R_{ab}(\lambda) = \mathbb{I}_\mathcal{N} \otimes \mathbb{I}_\mathcal{N} - \frac{\hbar\, \mathcal{P}_{ab}}{\lambda} \,, \tag{1.1}$$

where $\mathcal{P}_{ab}$ is the permutation operator

$$\mathcal{P}_{ab} = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes E_{ji} \tag{1.2}$$

and $\hbar$ is the free deformation parameter. It satisfies the following properties

*(i) Yang–Baxter equation* [33, 31, 32, 34, 35]

$$R_{ab}(\lambda_a - \lambda_b)\, R_{ac}(\lambda_a)\, R_{bc}(\lambda_b) = R_{bc}(\lambda_b)\, R_{ac}(\lambda_a)\, R_{ab}(\lambda_a - \lambda_b) \tag{1.3}$$

*(ii) Unitarity*

$$R_{ab}(\lambda) \ R_{ba}(-\lambda) = \zeta(\lambda) \, \mathbb{I}_{\mathcal{N}} \otimes \mathbb{I}_{\mathcal{N}} \ , \tag{1.4}$$

where $R_{ba}(\lambda) = \mathcal{P}_{ab} R_{ab}(\lambda) \mathcal{P}_{ab} = R_{ab}^{t_a t_b}(\lambda) = R_{ab}(\lambda)$ and

$$\zeta(\lambda) = \left(1 - \frac{\hbar}{\lambda}\right)\left(1 + \frac{\hbar}{\lambda}\right) \ . \tag{1.5}$$

It obeys $[A_a A_b, \ R_{ab}(\lambda)] = 0$ for $A \in End(\mathbb{C}^{\mathcal{N}})$.

The Yangian $\mathcal{Y}(gl(\mathcal{N}))$ [36] is the complex associative unital algebra with the generators $\{L_{ij}^{(n)} | 1 \leq i, j \leq \mathcal{N}, n \in \mathbb{Z}_{\geq 0}\}$ subject to the defining relations

$$[L_{ij}^{(r+1)} , \, L_{kl}^{(s)}] - [L_{ij}^{(r)} , \, L_{kl}^{(s+1)}] = L_{kj}^{(r)} \, L_{il}^{(s)} - L_{kj}^{(s)} \, L_{il}^{(r)} \ , \tag{1.6}$$

where $r, s \in \mathbb{Z}_{\geq 0}$ and $L_{ij}^{(0)} = \delta_{ij}$. These relations are encoded in a simple equation, called FRT exchange relation [37]

$$R_{ab}(\lambda_a - \lambda_b) \ \mathcal{L}_a(\lambda_a) \ \mathcal{L}_b(\lambda_b) = \mathcal{L}_b(\lambda_b) \ \mathcal{L}_a(\lambda_a) \ R_{ab}(\lambda_a - \lambda_b) \ , \tag{1.7}$$

where the generators are gathered in the following matrix (belonging to $End(\mathbb{C}^{\mathcal{N}}) \otimes \mathcal{Y}(gl(\mathcal{N}))[[\lambda^{-1}]]$)

$$\mathcal{L}(\lambda) = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes L_{ij}(\lambda) = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes \sum_{r \geq 0} \frac{\hbar^r}{\lambda^r} \, L_{ij}^{(r)} = \sum_{r \geq 0} \frac{\hbar^r}{\lambda^r} \, \mathcal{L}^{(r)} \ . \tag{1.8}$$

The quantum determinant qdet $\mathcal{L}(\lambda)$ is a formal series in $\lambda^{-1}$ with coefficients in $\mathcal{Y}(gl(\mathcal{N}))$ defined as follows

$$\text{qdet} \, \mathcal{L}(\lambda) = \sum_{\sigma \in \mathfrak{S}_{\mathcal{N}}} sgn(\sigma) \ L_{1,\sigma(1)}(\lambda - \hbar \mathcal{N} + \hbar) \cdots L_{\mathcal{N},\sigma(\mathcal{N})}(\lambda) \ , \tag{1.9}$$

where $\mathfrak{S}_{\mathcal{N}}$ is the permutation group of $\mathcal{N}$ indices. A well-known result (see e.g. [38]) establishes that the coefficients of qdet $\mathcal{L}(\lambda)$ are algebraically independent and generate the center of $\mathcal{Y}(gl(\mathcal{N}))$. There exists an equivalent definition of the quantum determinant which will be used in the following as well:

$$\text{qdet} \, \mathcal{L}(\lambda) \, A_{\mathcal{N}} = \mathcal{L}_{\mathcal{N}}(\lambda - \hbar \mathcal{N} + \hbar) \cdots \mathcal{L}_1(\lambda) \, A_{\mathcal{N}} \ . \tag{1.10}$$

where $A_{\mathcal{N}}$ is the antisymmetriser operator, a one-dimensional projector in $(\mathbb{C}^{\mathcal{N}})^{\otimes \mathcal{N}}$, i.e.

$$A_m(e_1 \otimes \cdots \otimes e_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} sgn(\sigma) \ e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(m)} \ . \tag{1.11}$$

For the study of spin chains, it is useful to introduce the following morphisms of $\mathcal{Y}(gl(\mathcal{N}))$:

**Inversion**

$$inv : \ \mathcal{L}(\lambda) \longmapsto \mathcal{L}^{-1}(\lambda) \tag{1.12}$$

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**Sign**

$$sg : \ \mathcal{L}(\lambda) \longmapsto \mathcal{L}(-\lambda) \,, \tag{1.13}$$

**Transposition**

$$t : \ \mathcal{L}_a(\lambda) \longmapsto \mathcal{L}_a^{t_a}(\lambda) \,, \tag{1.14}$$

where $^{t_a}$ is a generalised transposition (defined below) acting in space $a$ only.

**Shift**

$$s_a : \ \mathcal{L}(\lambda) \longmapsto \mathcal{L}(\lambda + a) \,, \ a \in \mathbb{C} \tag{1.15}$$

The three first mapping are idempotent algebra anti-morphisms, while the last one is an algebra automorphism.

The generalised transposition $^t$, which depends on a sign $\theta = \pm 1$, is related to the usual transposition $^T$ by, for any matrix $A$,

$$A^t = V^{-1} A^T V \ \ \text{where} \ \begin{cases} V = \text{antidiag}(1, 1, \ldots, 1) \,, & \text{for which} \ \ V^2 = \theta = 1 \\ \text{or} \\ V = \text{antidiag}\Big( \underbrace{1, \ldots, 1}_{\mathcal{N}/2}, \underbrace{-1, \ldots, -1}_{\mathcal{N}/2} \Big) \,, & \text{for which} \ \ V^2 = \theta = -1 \,. \end{cases} \tag{1.16}$$

The second case is forbidden when $\mathcal{N}$ is odd.

The elements of $\mathcal{L}^{-1}(\lambda)$ are computed in terms of $\mathcal{L}(\lambda)$ by

$$\mathcal{L}^{-1}(\lambda - \hbar\mathcal{N} + \hbar) = \big( \text{qdet} \, \mathcal{L}(\lambda) \big)^{-1} \mathcal{L}^*(\lambda) \,, \tag{1.17}$$

where $\mathcal{L}^*(\lambda)$ is the quantum comatrix, i.e. the entries $L_{ij}^*(\lambda)$ of $\mathcal{L}^*(\lambda)$ are $(-1)^{i+j}$ times the quantum determinants of the submatrices of $\mathcal{L}(\lambda)$ obtained by removing the $i^{th}$ column and $j^{th}$ row.

In the following, $\mathcal{L}(\lambda)$ will be denoted by $\mathcal{L}_{ap}(\lambda)$ since we deal with tensor products of Yangian. The index $a$ is the auxiliary space index (i.e. the space on which $E_{ij}$ acts) and $p$ is the quantum space index (i.e. the space on which $L_{ij}^{(r)}$ acts). Starting from local operators $\mathcal{L}_{ap}(\lambda)$ $(1 \leq p \leq \ell)$ acting on different spaces, one constructs a non-local algebraic object, the monodromy matrix

$$\mathcal{T}_a(\lambda) = \mathcal{L}_{a1}(\lambda) \, \mathcal{L}_{a2}(\lambda) \, \ldots \, \mathcal{L}_{a\ell}(\lambda) \in End(\mathbb{C}^\mathcal{N}) \otimes (\mathcal{Y}(gl(\mathcal{N})))^{\otimes \ell} \tag{1.18}$$

which satisfies the defining relations of the Yangian

$$R_{ab}(\lambda_a - \lambda_b) \, \mathcal{T}_a(\lambda_a) \, \mathcal{T}_b(\lambda_b) = \mathcal{T}_b(\lambda_b) \, \mathcal{T}_a(\lambda_a) \, R_{ab}(\lambda_a - \lambda_b) \,. \tag{1.19}$$

# 2 Algebraic transfer matrix for twisted Yangian

## 2.1 $K$ matrix

In order to study open spin chains with soliton non-preserving boundary conditions, we need to use subalgebras of Yangians called twisted Yangians, $\mathcal{Y}^{\pm}(\mathcal{N})$ [24]. The sign $+$ or $-$ allows us to choose between the two types of twisted Yangians (orthogonal or symplectic). First, we need to introduce numerical matrices, called $K$ matrices, which are solutions to the soliton non-preserving reflection equation[1] [24]

$$R_{ab}(\lambda_a - \lambda_b)\, K_a(\lambda_a)\, R_{ab}^{t_a}(-\lambda_a - \lambda_b - \hbar\rho)\, K_b(\lambda_b) =$$
$$K_b(\lambda_b)\, R_{ab}^{t_a}(-\lambda_a - \lambda_b - \hbar\rho)\, K_a(\lambda_a)\, R_{ba}(\lambda_a - \lambda_b)\,, \tag{2.1}$$

In this case, the $K$ matrix is interpreted as the reflection of a soliton on the boundary, coming back as an anti-soliton. The solutions of (2.1) have been classified in [21]:

**Proposition 2.1** *Any invertible solution of the soliton non-preserving reflection equation (2.1) is a constant matrix (up to a multiplication by a scalar function) such that $K^t = \varepsilon K$ with $\varepsilon = \pm 1$.*

In the following, we restrict ourselves to the case where $K_a$ is diagonal and is given by

$$K = diag(\zeta_1, \ldots, \zeta_{\mathcal{N}}) \qquad \text{with} \quad \zeta_{\overline{k}} = \varepsilon\, \zeta_k\,, \quad \zeta_k \neq 0 \qquad \text{and} \quad \varepsilon = \pm 1\,. \tag{2.2}$$

The variables $\zeta_k$ are $\left[\frac{\mathcal{N}+1}{2}\right]$ free parameters of the model.
Let us remark that when $\mathcal{N}$ is odd, one must take $\varepsilon = +1$ to ensure the invertibility of $K$ (and $\theta = +1$ in this case), while for $\mathcal{N}$ even, there are four choices ($\theta = \pm 1$, $\varepsilon = \pm 1$).

## 2.2 Twisted Yangians

The twisted Yangians are constructed as subalgebras of the Yangian $\mathcal{Y}(gl(\mathcal{N}))$. Starting from the generators $\mathcal{T}(\lambda)$ of $\mathcal{Y}(gl(\mathcal{N}))$ introduced in (1.8), we define

$$\mathcal{S}_a(\lambda) = \mathcal{T}_a(\lambda)\, K_a\, \mathcal{T}_a^{t_a}(-\lambda - \hbar\rho)\,. \tag{2.3}$$

$\mathcal{S}(\lambda)$ generates the algebra $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$ whose exchange relations[1] are given by

$$R_{ab}(\lambda_a - \lambda_b)\, \mathcal{S}_a(\lambda_a)\, R_{ab}^{t_a}(-\lambda_a - \lambda_b - \hbar\rho)\, \mathcal{S}_b(\lambda_b) =$$
$$\mathcal{S}_b(\lambda_b)\, R_{ab}^{t_a}(-\lambda_a - \lambda_b - \hbar\rho)\, \mathcal{S}_a(\lambda_a)\, R_{ba}(\lambda_a - \lambda_b)\,. \tag{2.4}$$

This relation is a direct consequence of (1.15), (1.14) and (2.1). The matrix $\mathcal{S}(\lambda)$ satisfies a supplementary symmetry relation

$$\mathcal{S}_a^{t_a}(\lambda) = \varepsilon\, \mathcal{S}_a(-\lambda - \hbar\rho) - \frac{\theta\,\hbar}{2\lambda + \hbar\rho}\, \big(\mathcal{S}_a(-\lambda - \hbar\rho) - \mathcal{S}_a(\lambda)\big)\,. \tag{2.5}$$

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[1]There exist different definitions of this relation depending upon the shift $\rho$ of the spectral parameter in $R_{ab}^{t_a}$.

As in equation (1.8), we define

$$\mathcal{S}(\lambda) = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes S_{ij}(\lambda) = \sum_{n=0}^{+\infty} \frac{\mathcal{S}^{(n)}}{\lambda^n} \; . \tag{2.6}$$

The commutation relations (2.4) and the symmetry relation (2.5) show that $\mathcal{S}^{(1)}$ generates a $so(\mathcal{N})$ (resp. a $sp(\mathcal{N})$) algebra when $\theta\varepsilon = 1$ (resp. $\theta\varepsilon = -1$), subalgebras in $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$ [24].

One can show that $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$ has a non-trivial center generated by the coefficients of the following series, the so-called Sklyanin determinant sdet $\mathcal{S}(\lambda)$ defined by

$$\mathcal{S}_{<a_{\mathcal{N}}\ldots a_1>}(\lambda)\, A_{\mathcal{N}} = A_{\mathcal{N}}\, \mathcal{S}_{<a_{\mathcal{N}}\ldots a_1>}(\lambda)\, A_{\mathcal{N}} = \text{sdet}\, \mathcal{S}(\lambda)\, A_{\mathcal{N}} \tag{2.7}$$

where

$$\mathcal{S}_{<a_{\mathcal{N}}\ldots a_1>}(\lambda) = \left( \overleftarrow{\prod_{2 \le k \le \mathcal{N}}} \mathcal{S}_{a_k}(\lambda_k)\, R_{a_k a_{k-1}}^{t_{a_k}}(-\lambda_k - \lambda_{k-1} - \hbar\rho) \ldots R_{a_k a_1}^{t_{a_k}}(-\lambda_k - \lambda_1 - \hbar\rho) \right) \mathcal{S}_{a_1}(\lambda_1) \tag{2.8}$$

and $\lambda_k = \lambda - \hbar(k-1)$. It satisfies the following relation

$$\text{sdet}\, \mathcal{S}(\lambda) = \text{sdet}\, K(\lambda) \quad \text{qdet}\, \mathcal{T}(\lambda) \quad \text{qdet}\, \mathcal{T}(-\lambda - \hbar(\rho - \mathcal{N} + 1))\,, \tag{2.9}$$

with

$$\text{sdet}\, K(\lambda) = \zeta_1\zeta_2\ldots\zeta_{\mathcal{N}}\, \frac{2\lambda - (\theta\varepsilon + 1)n\hbar + \hbar + \hbar\rho}{2\lambda - 2n\hbar + \hbar + \hbar\rho} \quad \text{and} \quad n = \left[ \frac{\mathcal{N}}{2} \right]\,. \tag{2.10}$$

## 2.3  Transfer matrix and symmetry of the model

The transfer matrix is defined by

$$s(\lambda) = tr_a \left( \mathcal{S}_a(\lambda) \right) = \sum_{i=1}^{\mathcal{N}} S_{ii}(\lambda) \tag{2.11}$$

and satisfies a crossing relation deduced from (2.5)

$$s(\lambda) = \frac{2\lambda\,\varepsilon + \hbar(\rho\,\varepsilon - \theta)}{2\lambda + \hbar(\rho - \theta)}\; s(-\lambda - \hbar\rho)\,. \tag{2.12}$$

This relation differs from the one given in [21], where the case $\varepsilon = +1$ is treated in the fundamental representation, because of the definitions (2.3) and (2.4) used here (see footnote 1). The commutation relations defining the twisted Yangian allows us to show

$$[s(\lambda), s(\mu)] = 0\,. \tag{2.13}$$

This commutation of the transfer matrix guarantees the integrability of the model characterized by the Hamiltonian which is any linear combination of the coefficients of $s(\lambda)$. The following proposition gives the symmetry of the model.

**Proposition 2.2** *For a generic $K$ matrix, the transfer matrix $s(\lambda)$ describing soliton non-preserving open spin chain models admits as symmetry algebra the direct sum of $\left[\frac{\mathcal{N}+1}{2}\right]$ copies of $\mathcal{G}$ where*

$$\mathcal{G} = \begin{cases} sp(2) & \text{algebras when} \quad \theta = -1 \text{ and } \varepsilon = 1 \\ so(2) & \text{algebras when} \quad \theta = 1 \text{ and } \varepsilon = 1 \\ U(1) & \text{algebras when } \varepsilon = -1. \end{cases} \tag{2.14}$$

*If $K$ contains $p$ equal parameters in the set $\{\zeta_j \mid 1 \leq j \leq \mathcal{N}/2\}$, then the corresponding symmetry subalgebra generated by the direct sum of the $p$ algebras $sp(2)$ (resp. $so(2)$ or $U(1)$) is enlarged to an $sp(2p)$ (resp. $so(2p)$ or $U(p)$) algebra.*
*In the particular case where $\mathcal{N} = 2n + 1$ and $\zeta_{n+1}$ is equal to the above $p$ parameters, the symmetry subalgebra generated by the direct sum of the $p + 1$ algebras $so(2)$ (resp. $U(1)$) is enlarged to an $so(2p + 1)$ (resp. $U(p + 1)$) algebra.*

**Proof:** Starting from (2.4), taking the trace in space $a$ and looking at the coefficient of $\lambda_b^{-1}$ and $E_{ij}$, one gets

$$\left[ s(\lambda), S_{ij}^{(1)} \right] = \hbar \left( \zeta_i - \zeta_j \right) \left( S_{ij}(\lambda) + (\mathcal{S}^t)_{ij}(\lambda) \right). \tag{2.15}$$

This shows that when $\zeta_i = \zeta_j$, $S_{ij}^{(1)}$ commutes with $s(\lambda)$, which leads to the different cases given in the proposition, taking into account the algebra generated by $\mathcal{S}^{(1)}$ (see above). $\blacksquare$
Let us remark that in proposition 2.2 all the symmetry algebras have the same rank $\left[\frac{\mathcal{N}+1}{2}\right]$.

# 3 Representations

In the following, it will be necessary to find the irreducible finite-dimensional representations of the twisted Yangians in order to construct soliton non-preserving open spin chains. For such a purpose, we reproduce the techniques used in [30] to classify these representations. The first step consists in constructing the representations of the local operators $\mathcal{L}_{ap}$. We then deal with the tensor product of these representations using (1.18). Finally, we obtain representation of the twisted Yangian by (2.3).

## 3.1 Evaluation representation of local operator

Let $\{e_{ij}\}$ be a basis of the Lie algebra $gl(\mathcal{N})$. The finite-dimensional irreducible representation of $gl(\mathcal{N})$, $M(\boldsymbol{\alpha})$, with highest weight $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{\mathcal{N}})$ and associated to the highest weight vector $v$ is characterized by

$$e_{kj}\, v = 0 \qquad , \quad 1 \leq k < j \leq \mathcal{N} \tag{3.1}$$

$$e_{kk}\, v = \alpha_k\, v \qquad , \quad 1 \leq k \leq \mathcal{N}, \tag{3.2}$$

where $\alpha_1, \ldots, \alpha_{\mathcal{N}} \in \mathbb{C}$.
The following algebra homomorphism from $\mathcal{Y}(gl(\mathcal{N}))$ to $\mathcal{U}(gl(\mathcal{N}))$ (universal enveloping algebra of

$gl(\mathcal{N}))$ [2]

$$L_{ij}(\lambda) \longmapsto \delta_{ij} - \frac{\hbar\, e_{ji}}{\lambda}\ , \tag{3.3}$$

allows us to build the evaluation representation $M_{\lambda+a}(\boldsymbol{\alpha})$ of $\mathcal{Y}(gl(\mathcal{N}))$ from $M(\boldsymbol{\alpha})$ satisfying

$$L_{jk}(\lambda)\, v = 0 \qquad,\quad 1 \le k < j \le \mathcal{N} \tag{3.4}$$

$$L_{kk}(\lambda)\, v = \left(1 - \frac{\hbar\,\alpha_k}{\lambda + a}\right) v \qquad,\quad 1 \le k \le \mathcal{N}\ . \tag{3.5}$$

It is important for the following to remark that the previous relations imply that the entries of the matrix $(\lambda + a)\mathcal{L}(\lambda)$ are analytical.

The representation $M_\lambda((1,0,\ldots,0))$ associated to the $gl(\mathcal{N})$ fundamental representation of $\mathcal{L}(\lambda)$ provides the $R$ matrix (1.1).

## 3.2 Representation of the Yangian

The evaluation representations of $\mathcal{L}(\lambda)$ allow us to build a representation of $\mathcal{T}(\lambda)$. Indeed, evaluating each of the local operator $\mathcal{L}_{an}(\lambda)$ in a representation $M_{\lambda+a_n}(\boldsymbol{\alpha^n})$ for $1 \le n \le \ell$, the tensor product built on

$$M_{\lambda+a_1}(\boldsymbol{\alpha^1}) \otimes \cdots \otimes M_{\lambda+a_\ell}(\boldsymbol{\alpha^\ell}) \tag{3.6}$$

provides a finite-dimensional representation for $\mathcal{T}(\lambda)$.

Denoting by $v^n$ the highest weight vector associated to $\boldsymbol{\alpha^n} = (\alpha_1^n, \ldots, \alpha_\mathcal{N}^n)$, the vector

$$v^+ = v^1 \otimes \cdots \otimes v^\ell \tag{3.7}$$

is the highest weight vector of the representation (3.6), i.e.

$$T_{jk}(\lambda)\, v^+ = 0 \qquad,\quad 1 \le k < j \le \mathcal{N} \tag{3.8}$$

$$T_{kk}(\lambda)\, v^+ = \prod_{n=1}^{\ell}\left(1 - \frac{\hbar\,\alpha_k^n}{\lambda + a_n}\right) v^+ \qquad,\quad 1 \le k \le \mathcal{N}\ . \tag{3.9}$$

For later convenience, we introduce the following polynomials, so-called Drinfeld polynomials:

$$P_k(\lambda) = \prod_{n=1}^{\ell}\left(\lambda + a_n - \hbar\,\alpha_k^n\right)\ . \tag{3.10}$$

We will be interested only in the irreducible finite-dimensional representations of the monodromy matrix. Indeed, when the representation is reducible, the Bethe ansatz does not give all the eigenvalues of the transfer matrix. There exists a necessary and sufficient criterion for a tensor product of Yangian representations to be irreducible [39, 29].

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[2]To be compatible with the pseudo-vacuum as usually defined in the study of spin chain models, the convention used here for the homomorphism differs from the one introduced in [30]. The link between the two conventions is provided by the Yangian automorphism $T(\lambda) \longmapsto T^T(-\lambda)$.

# 4 Spin chains with non preserving boundary conditions

## 4.1 Representations of $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$

The monodromy matrix used to construct open spin chains with non-preserving boundary conditions is constructed from the realization of the Yangian (1.18) and takes the following form

$$\mathcal{S}_a(\lambda) = \mathcal{L}_{a1}(\lambda) \ \ldots \ \mathcal{L}_{a\ell}(\lambda) \ K_a \ \mathcal{L}_{a\ell}^{t_a}(-\lambda - \hbar\rho) \ \ldots \ \mathcal{L}_{a1}^{t_a}(-\lambda - \hbar\rho) \ . \tag{4.1}$$

In order to study the representations of $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$, we start from the Yangian representations introduced in section 3. Let $M_\lambda(\boldsymbol{\alpha})$ be an evaluation representation of $\mathcal{L}(\lambda)$ with the highest weight vector $v$. We easily see that $v$ is the highest weight vector of $\mathcal{L}^t(\lambda)$ with

$$\overline{L}_{jk}(\lambda) \, v = L_{\overline{kj}}(\lambda) \, v = 0 \, , \qquad 1 \le k < j \le \mathcal{N} \tag{4.2}$$

$$\overline{L}_{kk}(\lambda) \, v = L_{\overline{kk}}(\lambda) \, v = \left(1 - \frac{\hbar \, \alpha_{\overline{k}}}{\lambda}\right) v \, , \qquad 1 \le k \le \mathcal{N} \, , \tag{4.3}$$

where $\overline{L}_{jk}(\lambda)$ are the matrix elements of the matrix $\mathcal{L}^t(\lambda)$ and the overlined indices $1 \le k \le \mathcal{N}$ are defined by

$$\overline{k} = \mathcal{N} + 1 - k \, . \tag{4.4}$$

It is known [30] that any finite-dimensional representation of $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$ is a highest weight representation. They can be constructed in the following way:

**Theorem 4.1** *Let us consider the Yangian highest weight representation* $M_{\lambda+a_1}(\boldsymbol{\alpha^1}) \otimes \ldots \otimes M_{\lambda+a_\ell}(\boldsymbol{\alpha^\ell})$ *with highest weight vector* $v^+ = v^1 \otimes \ldots \otimes v^\ell$. *We change the normalization in order to have analytical eigenvalues:*

$$\widehat{\mathcal{S}}_a(\lambda) = \prod_{i=1}^\ell (\lambda + a_i)(-\lambda - \hbar\rho + a_i) \ \mathcal{S}_a(\lambda) \qquad and \quad \widehat{s}(\lambda) = tr_a \widehat{\mathcal{S}}_a(\lambda) \, . \tag{4.5}$$

*Then the realization (4.1) generates a* $\mathcal{Y}^{\theta\varepsilon}(\mathcal{N})$ *highest weight representation, whose highest weight vector is also* $v^+$ *with*

$$\widehat{S}_{jk}(\lambda) \, v^+ = 0 \, , \qquad 1 \le k < j \le \mathcal{N} \tag{4.6}$$

$$\widehat{S}_{kk}(\lambda) \, v^+ = \zeta_k \sigma_k(\lambda) \, v^+ \, , \qquad 1 \le k \le \frac{\mathcal{N}+1}{2} \, , \tag{4.7}$$

$$\widehat{S}_{kk}(\lambda) \, v^+ = \frac{\zeta_k}{2\lambda + \hbar\rho} \Big( (2\lambda + \hbar(\rho - \theta\varepsilon)) \sigma_k(\lambda) + \theta\varepsilon\hbar \ \sigma_{\overline{k}}(\lambda) \Big) v^+ \, , \qquad \frac{\mathcal{N}}{2} + 1 \le k \le \mathcal{N} \tag{4.8}$$

*where*

$$\sigma_k(\lambda) = P_k(\lambda) P_{\overline{k}}(-\lambda - \hbar\rho) \, . \tag{4.9}$$

*Let us recall that the* $\zeta_i$ *are the elements of the diagonal* $K$ *matrix satisfying eq. (2.2).*

**Proof:** The equalities (4.6) and (4.7) are a consequence of (4.2) and (4.3) as well as of a direct computation using the Yangian commutation relations. The last relation (4.8) is computed from the previous ones using (2.5). ∎

Remark that the functions $\sigma_k(\lambda)$ obey the crossing relations

$$\sigma_{\bar{k}}(-\lambda - \hbar\rho) = \sigma_k(\lambda)\,. \tag{4.10}$$

For a given representation, the Sklyanin determinant is given by

$$\mathrm{sdet}\,\widehat{\mathcal{S}}(\lambda) = \mathrm{sdet}\,K(\lambda)\,\prod_{k=1}^{\mathcal{N}}\sigma_k(\lambda - \hbar(\mathcal{N}-k))\,. \tag{4.11}$$

## 4.2 An automorphism of the twisted Yangian

The following map

$$
\begin{aligned}
\mathcal{S}(\lambda) \longrightarrow \check{\mathcal{S}}(\lambda) \;&=\; \frac{\mathrm{sdet}\,\mathcal{S}(-\lambda + \hbar(\mathcal{N}/2 - 1 - \rho))}{\mathrm{sdet}\,K(-\lambda + \hbar(\mathcal{N}/2 - 1 - \rho))}\;\; \mathcal{S}^{-1}(-\lambda - \hbar(\mathcal{N}/2 + \rho)) \tag{4.12}\\
&=\; \mathrm{qdet}\,\mathcal{T}(-\lambda + \hbar(\mathcal{N}/2 - 1 - \rho))\;\; \mathrm{qdet}\,\mathcal{T}(\lambda + \hbar\mathcal{N}/2)\\
&\quad\times \big(\mathcal{T}(\lambda + \hbar\mathcal{N}/2)^t\big)^{-1}\,K^{-1}\,\mathcal{T}^{-1}(-\lambda - \hbar(\mathcal{N}/2 + \rho)) \tag{4.13}
\end{aligned}
$$

is an automorphism of the twisted Yangien [30]. In order to deal with analytical entries, we change the normalization of $\check{\mathcal{S}}(\lambda)$ as follows:

$$\widetilde{\mathcal{S}}(\lambda) = \prod_{k=1}^{\mathcal{N}-1}\left(\prod_{i=1}^{\ell}\Big(\lambda + a_i - \hbar(\mathcal{N}/2 - k)\Big)\Big(-\lambda + a_i - \hbar(\mathcal{N}/2 - k + \rho)\Big)\right)\check{\mathcal{S}}(\lambda) \tag{4.14}$$

One can show that the vector $v^+$ is also a highest weight vector for $\widetilde{S}(\lambda)$ with

$$\widetilde{S}_{jk}(\lambda)\,v^+ = 0\,, \qquad \text{for}\quad 1 \le k < j \le \mathcal{N} \tag{4.15}$$

$$\widetilde{S}_{kk}(\lambda)\,v^+ = \frac{1}{\zeta_k}\,\widetilde{\sigma}_k(\lambda)\,v^+\,, \qquad \text{for}\quad 1 \le k \le \frac{\mathcal{N}+1}{2} \tag{4.16}$$

and for $\frac{\mathcal{N}}{2} + 1 \le k \le \mathcal{N}$,

$$\widetilde{S}_{kk}(\lambda)\,v^+ = \frac{\theta\hbar}{2\lambda + \hbar\rho}\widetilde{S}_{\bar{k}\bar{k}}(\lambda)\,v^+ + \frac{2\varepsilon\lambda + \hbar(\varepsilon\rho - \theta)}{2\lambda + \hbar\rho}\widetilde{S}_{\bar{k}\bar{k}}(-\lambda - \hbar\rho)\,v^+\,. \tag{4.17}$$

where

$$\widetilde{\sigma}_k(\lambda) = \prod_{j=1}^{k-1}\sigma_j(-\lambda - \hbar(\mathcal{N}/2 + \rho - j))\,\prod_{j=k+1}^{\mathcal{N}}\sigma_j(-\lambda - \hbar(\mathcal{N}/2 + \rho - j + 1))\,. \tag{4.18}$$

The equality (4.16) is computed similarly to (4.7). The relation (4.17) follows from the crossing relation (2.5) satisfied by $\check{\mathcal{S}}(\lambda)$. Note that the functions $\widetilde{\sigma}_k(\lambda)$ also obey the crossing relation

$$\widetilde{\sigma}_{\bar{k}}(-\lambda - \hbar\rho) = \widetilde{\sigma}_k(\lambda)\,. \tag{4.19}$$

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The transfer matrix $\widetilde{s}(\lambda) = tr_a \widetilde{\mathcal{S}}_a(\lambda)$ satisfies the commutation relations

$$[\widetilde{s}(\lambda), \widetilde{s}(\mu)] = 0 \qquad , \quad [\widetilde{s}(\lambda), \widehat{s}(\mu)] = 0 \tag{4.20}$$

and the crossing relation

$$\widetilde{s}(\lambda) = \frac{2\lambda\,\varepsilon + \hbar(\rho\,\varepsilon - \theta)}{2\lambda + \hbar(\rho - \theta)} \; \widetilde{s}(-\lambda - \hbar\rho) \ . \tag{4.21}$$

The Liouville contraction [24] allows us to find a constraint between $\widehat{s}(u)$ and $\widetilde{s}(u)$:

$$\widetilde{s}(u)\widehat{s}(u - \hbar\mathcal{N}/2) = \left(1 + \frac{\hbar\theta}{2u - \hbar(\mathcal{N} - \rho)}\right)\left(\varepsilon - \frac{\hbar\theta}{2u + \hbar\rho}\right)\prod_{k=1}^{\mathcal{N}}\sigma_{\overline{k}}(u + \hbar(\mathcal{N}/2 + k)) + s_{\mathsf{f}}(u) \ . \tag{4.22}$$

This relation is computed by using the following equality (see ref. [24])

$$\frac{Q_{12}}{\mathcal{N}}R_{12}(-2u + \hbar(\mathcal{N} - \rho))\widetilde{\mathcal{S}}_1(u)R_{12}(2u + \hbar\rho)\widehat{\mathcal{S}}_2(u - \hbar\mathcal{N}/2)$$

$$= \frac{Q_{12}}{\mathcal{N}}\left(1 + \frac{\hbar\theta}{2u - \hbar(\mathcal{N} - \rho)}\right)\left(\varepsilon - \frac{\hbar\theta}{2u + \hbar\rho}\right)\prod_{k=1}^{N}\sigma_{\overline{k}}(u + \hbar(\mathcal{N}/2 + k)) \tag{4.23}$$

where $Q_{12} = \mathcal{P}_{12}^{t_1}$ and $Q_{12}/\mathcal{N}$ is a one-dimensional projector.

## 4.3 Analytical Bethe ansatz

### 4.3.1 Pseudo-vacuum

As in the case of the closed spin chains, the first step of the analytical Bethe Ansatz consists in finding a particular eigenvalue of the transfer matrix $s(\lambda)$. This eigenvalue is computed thanks to the highest weight vector $v^+$. Indeed, one gets

$$\widehat{s}(\lambda)\,v^+ = \sum_{k=1}^{\mathcal{N}}\widehat{S}_{kk}(\lambda)v^+ = \Lambda^0(\lambda)\,v^+ \tag{4.24}$$

where

$$\Lambda^0(\lambda) = \sum_{k=1}^{\mathcal{N}}g_k(\lambda)\,\sigma_k(\lambda) \ . \tag{4.25}$$

The functions $g_k(\lambda)$ depend on the boundary and are given by

$$g_k(\lambda) = \begin{cases} \zeta_k\,\dfrac{2\lambda + \hbar(\rho + \theta)}{2\lambda + \hbar\rho} & , \quad 1 \le k \le \dfrac{\mathcal{N}}{2} \ , \\[2mm] \zeta_k & \text{if}\quad \mathcal{N}\ \text{is odd and } k = \frac{\mathcal{N}+1}{2} \\[2mm] \zeta_k\,\dfrac{2\lambda + \hbar(\rho - \theta\varepsilon)}{2\lambda + \hbar\rho} & , \quad \dfrac{\mathcal{N}}{2} + 1 \le k \le \mathcal{N} \end{cases} \tag{4.26}$$

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while the functions $\sigma_k(\lambda)$ depend on the choice of the representation and are given by (4.9). Note that, although $g_k(\lambda)$ has a pole at $\lambda = -\hbar\rho/2$, the residue $res_{\lambda=-\hbar\rho/2}\Lambda^0(\lambda)$ vanishes, so that the eigenvalue is indeed analytical.

The functions $g_k(\lambda)$ satisfies the following relation

$$g_k(\lambda) = \frac{2\lambda\varepsilon + \hbar(\rho\varepsilon - \theta)}{2\lambda + \hbar(\rho - \theta)} \; g_{\overline{k}}(-\lambda - \hbar\rho) \; . \tag{4.27}$$

### 4.3.2 Dressing functions and fusion procedure

The analytical Bethe ansatz states that all the eigenvalues of $\widehat{s}(\lambda)$ can be written as

$$\Lambda(\lambda) = \sum_{k=1}^{\mathcal{N}} g_k(\lambda) \; \sigma_k(\lambda) \; D_k(\lambda) \; , \tag{4.28}$$

where the dressing functions $D_k(\lambda)$ are rational functions and need to be determined. The crossing relations (2.12), (4.10) and (4.27) imply that

$$D_k(\lambda) = D_{\overline{k}}(-\lambda - \hbar\rho) \qquad \text{for} \quad 1 \le k \le \mathcal{N} \; . \tag{4.29}$$

This latter relation is sufficient to prove that the residue of $\Lambda(\lambda)$ at $\lambda = -\hbar\rho/2$ vanishes.

In order to further constrain the dressing functions, we use the Sklyanin determinant to fuse $\mathcal{N}$ auxiliary spaces according to $[\mathcal{N}]^{\otimes \mathcal{N}} = [1] \oplus \dots$. We start with the decomposition

$$\mathcal{K}^+(\lambda)\,\mathcal{S}_{<a_{\mathcal{N}}\dots a_1>}(\lambda) = \mathcal{K}^+(\lambda)\,\mathcal{S}_{<a_{\mathcal{N}}\dots a_1>}(\lambda)\,A_{\mathcal{N}} + \mathcal{K}^+(\lambda)\,\mathcal{S}_{<a_{\mathcal{N}}\dots a_1>}(\lambda)\,(1 - A_{\mathcal{N}}) \tag{4.30}$$

where $\mathcal{S}_{<a_{\mathcal{N}}\dots a_1>}(\lambda)$ has been introduced in (2.8),

$$\mathcal{K}^+(\lambda) = \overrightarrow{\prod_{2 \le k \le \mathcal{N}}} \left( R_{a_k a_1}^{t_{a_k}}(\lambda_k + \lambda_1 + \hbar\rho) \cdots R_{a_k a_{k-1}}^{t_{a_k}}(\lambda_k + \lambda_{k-1} + \hbar\rho) \right) \tag{4.31}$$

and $\lambda_k = \lambda - \hbar(k-1)$. Using the property (2.7) and taking the trace over the auxiliary spaces $a_1,\dots,$ $a_{\mathcal{N}}$, we get after some calculation[3]

$$\frac{2\lambda + \hbar(\rho - 2\mathcal{N} + 2)}{2\lambda + \hbar(\rho - \mathcal{N} + 1)} \prod_{j=1}^{\mathcal{N}-1} \frac{2\lambda + \hbar(\rho - 2j + 2)}{2\lambda + \hbar(\rho - 2j + 1)} \widehat{s}(\lambda_{\mathcal{N}}) \cdots \widehat{s}(\lambda_1)$$

$$= \frac{2\lambda + \hbar\left(\rho - (3 - \theta)\,n + 1\right)}{2\lambda + \hbar(\rho - 2n + 1)} \, \text{sdet}\,\widehat{\mathcal{S}}(\lambda) + s_{\mathsf{f}}(\lambda) \tag{4.32}$$

where $n = \left[\dfrac{\mathcal{N}}{2}\right]$. Applying relation (4.32) on an $\widehat{s}(\lambda)$ eigenvector, we obtain using relation (2.9) the fusion relation for the soliton non-preserving dressing functions:

$$D_1(\lambda - \hbar\mathcal{N} + \hbar)\dots D_{\mathcal{N}}(\lambda) = 1 \; . \tag{4.33}$$

---

[3]The introduction of $\mathcal{K}_+(\lambda)$ in the process is essential to show that the l.h.s. is a function of the transfer matrix solely.

### 4.3.3 Dressing functions for $\widetilde{s}(\lambda)$

The entries of $\widetilde{S}(\lambda)$ are analytical. Therefore, we can use the analytical Bethe ansatz to compute the eigenvalues of $\widetilde{s}$. One gets

$$\widetilde{s}(\lambda)\, v^+ = \sum_{k=1}^{\mathcal{N}} \widetilde{S}_{kk}(\lambda) v^+ = \widetilde{\Lambda}^0(\lambda)\, v^+ \tag{4.34}$$

where

$$\widetilde{\Lambda}^0(\lambda) = \sum_{k=1}^{\mathcal{N}} \widetilde{g}_k(\lambda)\, \widetilde{\sigma}_k(\lambda)\,. \tag{4.35}$$

The functions $\widetilde{g}_k(\lambda)$ depend on the boundary and are given by

$$\widetilde{g}_k(\lambda) = \frac{1}{\zeta_k^2}\, g_k(\lambda) \qquad \text{for} \quad 1 \le k \le \mathcal{N} \tag{4.36}$$

and the functions $\widetilde{\sigma}_k(\lambda)$ are given by relation (4.18). Let us remark, *en passant*, that we have the relations

$$\frac{\widetilde{\sigma}_{k+1}(\lambda)}{\widetilde{\sigma}_k(\lambda)} = \frac{\sigma_k(-\lambda - \hbar(\mathcal{N}/2 + \rho - k))}{\sigma_{k+1}(-\lambda - \hbar(\mathcal{N}/2 + \rho - k))} \tag{4.37}$$

which allows us to relate the irreducibility criterion for $\widetilde{S}(\lambda)$ to the ones of $S(\lambda)$, in accordance with the automorphism (4.12).

At this point, using the same ansatz than in the case of $\Lambda(\lambda)$, the above eigenvalue can be dressed to obtain all the eigenvalues of $\widetilde{S}(\lambda)$

$$\widetilde{\Lambda}(\lambda) = \sum_{k=1}^{\mathcal{N}} \widetilde{g}_k(\lambda)\, \widetilde{\sigma}_k(\lambda) \widetilde{D}_k(\lambda)\,. \tag{4.38}$$

Using the crossing relations (4.21), (4.27) and (4.19), the dressing functions satisfy the following crossing:

$$\widetilde{D}_k(\lambda) = \widetilde{D}_{\overline{k}}(-\lambda - \hbar\rho)\,. \tag{4.39}$$

Through a fusion procedure with the Sklyanin determinant of $\widetilde{S}$, the following constraint is obtained:

$$\widetilde{D}_1(\lambda - \mathcal{N}\hbar + \hbar)\ldots\widetilde{D}_{\mathcal{N}}(\lambda) = 1\,. \tag{4.40}$$

Finally, from (4.22) one gets

$$\widetilde{D}_{\mathcal{N}}(\lambda) D_1(\lambda - \hbar\mathcal{N}/2) = 1\,. \tag{4.41}$$

### 4.3.4 Constraints for the dressing functions

Guided by the results given in [20, 21] for SNP spin chains in the fundamental representation, we assume that the dressing functions are rational functions of the form

$$D_k(\lambda) = \prod_{j=1}^{M^{(k-1)}} \frac{\lambda + u_j^{(k-1)}}{\lambda - \lambda_j^{(k-1)} - \frac{\hbar\,(k-1)}{2}} \frac{\lambda + v_j^{(k-1)}}{\lambda + \lambda_j^{(k-1)} - \frac{\hbar\,(k-1)}{2}} \prod_{j=1}^{M^{(k)}} \frac{\lambda + w_j^{(k)}}{\lambda - \lambda_j^{(k)} - \frac{\hbar\,k}{2}} \frac{\lambda + x_j^{(k)}}{\lambda + \lambda_j^{(k)} - \frac{\hbar\,k}{2}} , \quad (4.42)$$

where $M^{(0)} = M^{(\mathcal{N})} = 0$ and $u_j^{(k)}$, $v_j^{(k)}$, $w_j^{(k)}$ and $x_j^{(k)}$ are coefficients to be determined. A similar assumption is made for the form of $\widetilde{D}_k(\lambda)$.

Let us remark that the above form is also dictated by the SNP reflection equation which shows that the exchange relations for the twisted Yangian generators always imply both $R(\lambda - \mu)$ and $R(\lambda + \mu)$. Having in mind the algebraic Bethe ansatz construction for the transfer matrix eigenvectors, we may conclude that the corresponding eigenvalues consist of terms of the form $f(\lambda - \lambda_j)\,f(\lambda + \lambda_j)$, hence the form (4.42) assumed here.

Imposing the constraints (4.29) and (4.33) for the dressing functions $D(\lambda)$, the constraints (4.39) and (4.40) for the dressing functions $\widetilde{D}(\lambda)$ and finally the constraint (4.41) between the two types of dressing functions, we obtain sufficient constraints to determine $u_j^{(k)}$, $v_j^{(k)}$, $w_j^{(k)}$ and $x_j^{(k)}$ (and also the $\widetilde{D}(\lambda)$ parameters) in terms of $\lambda_j^{(k)}$. The dressing functions become, for $1 \le k \le [\frac{\mathcal{N}-1}{2}]$,

$$D_k(\lambda) = \prod_{j=1}^{M^{(k-1)}} \frac{\lambda - \lambda_j^{(k-1)} - \frac{\hbar\,(k+1)}{2}}{\lambda - \lambda_j^{(k-1)} - \frac{\hbar\,(k-1)}{2}} \frac{\lambda + \lambda_j^{(k-1)} - \frac{\hbar\,(k+1)}{2}}{\lambda + \lambda_j^{(k-1)} - \frac{\hbar\,(k-1)}{2}} \prod_{j=1}^{M^{(k)}} \frac{\lambda - \lambda_j^{(k)} - \frac{\hbar\,(k-2)}{2}}{\lambda - \lambda_j^{(k)} - \frac{\hbar\,k}{2}} \frac{\lambda + \lambda_j^{(k)} - \frac{\hbar\,(k-2)}{2}}{\lambda + \lambda_j^{(k)} - \frac{\hbar\,k}{2}} . (4.43)$$

When $\mathcal{N} = 2n$ and $\rho \ne -\mathcal{N}/2$, we have the following particular form for $D_n(\lambda)$

$$
\begin{aligned}
D_n(\lambda) &= \prod_{j=1}^{M^{(n-1)}} \frac{\lambda - \lambda_j^{(n-1)} - \frac{\hbar\,(n+1)}{2}}{\lambda - \lambda_j^{(n-1)} - \frac{\hbar\,(n-1)}{2}} \frac{\lambda + \lambda_j^{(n-1)} - \frac{\hbar\,(n+1)}{2}}{\lambda + \lambda_j^{(n-1)} - \frac{\hbar\,(n-1)}{2}} \\
&\times \prod_{j=1}^{M^{(n)}} \frac{\lambda - \lambda_j^{(n)} - \frac{\hbar\,(n-2)}{2}}{\lambda - \lambda_j^{(n)} - \frac{\hbar\,n}{2}} \frac{\lambda + \lambda_j^{(n)} - \frac{\hbar\,(n-2)}{2}}{\lambda + \lambda_j^{(n)} - \frac{\hbar\,n}{2}} \frac{\lambda - \lambda_j^{(n)} + \frac{\hbar\,(n+2+2\rho)}{2}}{\lambda - \lambda_j^{(n)} + \frac{\hbar\,(n+2\rho)}{2}} \frac{\lambda + \lambda_j^{(n)} + \frac{\hbar\,(n+2+2\rho)}{2}}{\lambda + \lambda_j^{(n)} + \frac{\hbar\,(n+2\rho)}{2}}
\end{aligned}
\quad (4.44)
$$

whereas for $\mathcal{N} = 2n + 1$, we have the following particular form for $D_{n+1}(\lambda)$

$$D_{n+1}(\lambda) = \prod_{j=1}^{M^{(n)}} \frac{\lambda - \lambda_j^{(n)} - \frac{\hbar\,(n+2)}{2}}{\lambda - \lambda_j^{(n)} - \frac{\hbar\,n}{2}} \frac{\lambda + \lambda_j^{(n)} - \frac{\hbar\,(n+2)}{2}}{\lambda + \lambda_j^{(n)} - \frac{\hbar\,n}{2}} \frac{\lambda - \lambda_j^{(n)} + \frac{\hbar\,(n+2+2\rho)}{2}}{\lambda - \lambda_j^{(n)} + \frac{\hbar\,(n+2\rho)}{2}} \frac{\lambda + \lambda_j^{(n)} + \frac{\hbar\,(n+2+2\rho)}{2}}{\lambda + \lambda_j^{(n)} + \frac{\hbar\,(n+2\rho)}{2}} \quad (4.45)$$

The other dressing functions, $D_k(\lambda)$ for $[\frac{\mathcal{N}+1}{2}] < k \le \mathcal{N}$, are determined by the crossing relations (4.29) and (4.39).

Note that when $\mathcal{N} = 2n$ and $\rho = -\mathcal{N}/2$, one has to modify the form (4.44). Indeed, in that case, the second line of (4.44) is a square, which has to be omitted. This leads to the simpler form:

$$D_n(\lambda) = \prod_{j=1}^{M^{(n-1)}} \frac{\lambda - \lambda_j^{(n-1)} - \frac{\hbar\,(n+1)}{2}}{\lambda - \lambda_j^{(n-1)} - \frac{\hbar\,(n-1)}{2}} \frac{\lambda + \lambda_j^{(n-1)} - \frac{\hbar\,(n+1)}{2}}{\lambda + \lambda_j^{(n-1)} - \frac{\hbar\,(n-1)}{2}} \prod_{j=1}^{M^{(n)}} \frac{\lambda - \lambda_j^{(n)} - \frac{\hbar\,(n-2)}{2}}{\lambda - \lambda_j^{(n)} - \frac{\hbar\,n}{2}} \frac{\lambda + \lambda_j^{(n)} - \frac{\hbar\,(n-2)}{2}}{\lambda + \lambda_j^{(n)} - \frac{\hbar\,n}{2}}$$

$$(4.46)$$

### 4.3.5   Bethe equations

By imposing that the vanishing of the $\lambda = \lambda_p^{(k)} + \frac{\hbar k}{2}$ residue of $\Lambda(\lambda)$, we obtain a supplementary constraint between the parameters $\lambda_j^{(n)}$ called Bethe equations. For $1 \leq k < \left[\frac{\mathcal{N}+1}{2}\right]$, these equations read

$$\frac{g_k(\lambda_p^{(k)} + \frac{\hbar k}{2})\sigma_k(\lambda_p^{(k)} + \frac{\hbar k}{2})}{g_{k+1}(\lambda_p^{(k)} + \frac{\hbar k}{2})\sigma_{k+1}(\lambda_p^{(k)} + \frac{\hbar k}{2})} = - \prod_{j=1}^{M^{(k-1)}} \widehat{e}_{-1}(\lambda_p^{(k)}, \lambda_j^{(k-1)}) \prod_{j=1}^{M^{(k)}} \widehat{e}_2(\lambda_p^{(k)}, \lambda_j^{(k)}) \prod_{j=1}^{M^{(k+1)}} \widehat{e}_{-1}(\lambda_p^{(k)}, \lambda_j^{(k+1)})$$

$$(4.47)$$

where

$$\widehat{e}_x(\lambda, \mu) = \frac{\lambda - \mu - \frac{\hbar x}{2}}{\lambda - \mu + \frac{\hbar x}{2}} \frac{\lambda + \mu - \frac{\hbar x}{2}}{\lambda + \mu + \frac{\hbar x}{2}} \tag{4.48}$$

The last equation ($\lambda = \lambda_p^{(n)} + \frac{\hbar n}{2}$ residue) depends on the parity of $\mathcal{N}$ and on the choice of $\rho$. For $\mathcal{N} = 2n$ and $\rho \neq -\mathcal{N}/2$, this equation reads

$$\frac{g_n(\lambda_p^{(n)} + \frac{\hbar n}{2})\sigma_n(\lambda_p^{(n)} + \frac{\hbar n}{2})}{g_{n+1}(\lambda_p^{(n)} + \frac{\hbar n}{2})\sigma_{n+1}(\lambda_p^{(n)} + \frac{\hbar n}{2})} =$$

$$- \prod_{j=1}^{M^{(n-1)}} \widehat{e}_{-1}(\lambda_p^{(n)}, \lambda_j^{(n-1)})\widehat{e}_{-1}(\lambda_p^{(n)} + \hbar(n+\rho), \lambda_j^{(n-1)}) \prod_{j=1}^{M^{(n)}} \widehat{e}_2(\lambda_p^{(n)}, \lambda_j^{(n)})\widehat{e}_2(\lambda_p^{(n)} + \hbar(n+\rho), \lambda_j^{(n)}) \quad (4.49)$$

In the particular case where $\rho = -\frac{\mathcal{N}}{2}$, this equation reduces to

$$\frac{g_n(\lambda_p^{(n)} + \frac{\hbar n}{2})\sigma_n(\lambda_p^{(n)} + \frac{\hbar n}{2})}{g_{n+1}(\lambda_p^{(n)} + \frac{\hbar n}{2})\sigma_{n+1}(\lambda_p^{(n)} + \frac{\hbar n}{2})} = - \prod_{j=1}^{M^{(n-1)}} \widehat{e}_{-1}(\lambda_p^{(n)}, \lambda_j^{(n-1)})^2 \prod_{j=1}^{M^{(n)}} \widehat{e}_2(\lambda_p^{(n)}, \lambda_j^{(n)}) \tag{4.50}$$

While for $\mathcal{N} = 2n + 1$, it writes

$$\frac{g_n(\lambda_p^{(n)} + \frac{\hbar n}{2})\sigma_n(\lambda_p^{(n)} + \frac{\hbar n}{2})}{g_{n+1}(\lambda_p^{(n)} + \frac{\hbar n}{2})\sigma_{n+1}(\lambda_p^{(n)} + \frac{\hbar n}{2})} = - \prod_{j=1}^{M^{(n-1)}} \widehat{e}_{-1}(\lambda_p^{(n)}, \lambda_j^{(n-1)})$$

$$\times \prod_{j=1}^{M^{(n)}} \widehat{e}_2(\lambda_p^{(n)}, \lambda_j^{(n)})\,\widehat{e}_{-1}(\lambda_p^{(n)} + \hbar\frac{\mathcal{N} + 2\rho}{2}, \lambda_j^{(n)}) \tag{4.51}$$

## 5   Discussion

Having specified the spectrum and Bethe ansatz equations for rational periodic and open spin chains in [29] and in this article, the next natural step is the derivation of the spectrum and the Bethe ansatz equations for the $q$-deformed case, for both periodic and open boundaries. This may be achieved by employing the analytical Bethe ansatz techniques, however the ultimate goal is the formulation of a generic algebraic Bethe ansatz method for deriving not only the spectrum and Bethe ansatz for any

irreducible representation of e.g $\mathcal{Y}(gl_\mathcal{N}))$, $U_q(gl_\mathcal{N})$, but also finding the corresponding eigenvectors. Such a process will exclusively rely on the exchange relations emerging from the Yang–Baxter or reflection equation, depending on the choice of the boundary conditions. This project is of great mathematical and physical relevance, and is under current investigation.

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